7 Three-Dimensional Coordinate Geometry

Before we lead on to a study of the graphical display of objects in three-dimensional space, we first have to come to terms with the three-dimensional Cartesian coordinate geometry. As in two-dimensional space, we arbitrarily fix a point in the space, named the *coordinate origin* (or or *origin* for short). We then imagine three mutually perpendicular lines through this point, each line going off to infinity in both directions. These are the x-axis, the y-axis and the z-axis. Each axis is thought to have a positive and a negative half, both starting at the origin; that is, distances measured from the origin along the axis are positive on one side and negative on the other. We may think of the x-axis and y-axis in the same way as we did for two-dimensional space, both lying on the page of this book say, the positive x-axis 'horizontal' and to the right of the origin, and the positive y-axis 'vertical' and above the origin. This just leaves the position of the z-axis: it has to be perpendicular to the page (since it is perpendicular to both the x-axis and the y-axis). The positive z-axis can be into the page (the so-called *left-handed triad* ofaxes) or out of the page (the *right-handed triad*). In this book we always use the left-handed triad notation. What we say in the remainder of the book, using left-handed axes, has its equivalent in the right-handed system - it does not matter which notation you finally decide to use as long as you are consistent.

We specify a general point $\boldsymbol{p}$ in space by a coordinate triple or vector (X, Y, Z), where the individual coordinate values are the perpendicular projections of the point on to the respective x-axis, .y-axis and z-axis. By projection we mean the unique point on the specified axis such that a line from that point to $\boldsymbol{p}$ is perpendicular to that axis.

Initially there are two operations we need to consider for three-dimensional vectors.Suppose we have two vectors, $\boldsymbol{p}_1 \equiv (x_1, y_1, z_1)$ and $\boldsymbol{p}_2 \equiv (x_2, y_2, z_2)$ then

scalar multiple: we multiply the three individual coordinate values by a scalar number k

$$k\boldsymbol{p}_1 = (k \times x_1, k \times y_1, k \times z_1)$$

vector addition: we add the x-coordinates together, then the y-coordinates and finally the z-coordinates to form a new vector

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 \equiv (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

Definition of a Straight Line

A *straight line* in three-dimensional space that passes through two points such as $\boldsymbol{p}_1 \equiv (x_1, y_1, z_1)$ and $\boldsymbol{p}_2 \equiv (x_2, y_2, z_2)$ is the next object to be defined. We may do this by describing the coordinates of a general point $\boldsymbol{p} \equiv (x, y, z)$

$$(x - x_1) \times (y_2 - y_1) = (y - y_1) \times (x_2 - x_1)$$
$$(y - y_1) \times (z_2 - z_1) = (z - z_1) \times (y_2 - y_1)$$
$$(z - z_1) \times (x_2 - x_1) = (x - x_1) \times (z_2 - z_1)$$

Although these are three equations in three unknowns, we shall see that they are inter-related (or so-called *linearly dependent*) and so there is no unique solution (that is natural since we are generating a general point on the line, not just one point). These equations enable us to calculate two of the coordinates in terms of a third (see example 7.1).

As with two dimensions, this is not the only way of representing a line, in fact the second way we introduce is possibly more useful. The general point on the line is represented as a vector that is dependent on only one real number μ, and is given as the vector sum of two scalar multiples ofvectors:

$$\boldsymbol{p}(\mu) \equiv (1 - \mu)\boldsymbol{p}_1 + \mu\boldsymbol{p}_2 \quad \text{where} \quad -\infty < \mu < \infty$$

That is

$$\boldsymbol{p}(\mu) \equiv ((1 - \mu) \times x_1 + \mu \times x_2, (1 - \mu) \times \mathrm{y}_1 + \mu \times y_2, (1 - \mu) \times z_1 + \mu \times z_2)$$

This form is exactly equivalent to the two-dimensional parametric form of a line that we saw in chapter 3. Here we place μ in brackets after $\boldsymbol{p}$ to demonstrate the dependence of $\boldsymbol{p}$ on μ; however, when this concept has been fully investigated, then (μ) will be ignored. Note that when $\mu = 0$ the equation returns point $\boldsymbol{p}_1$ and when $\mu = 1$ it gives point $\boldsymbol{p}_2$.

We may rewrite this vector expression as

$$\boldsymbol{p}(\mu) = \boldsymbol{p}_1 + \mu(\boldsymbol{p}_2 - \boldsymbol{p}_1)$$

Like its counterpart in two dimensions, $\boldsymbol{p}_1$ is called a base vector and $(\boldsymbol{p}_2 - \boldsymbol{p}_1)$ a directional vector. Again we see the dual interpretation of a vector. A vector may be used to specify a point uniquely in three-dimensional space, or it may be considered as a general direction, namely any line parallel to the line that joins the origin to the vector (considered as a point). We can move along a line in one of two directions, so we say that the direction from the origin to the point has a positive sense, and the direction from the point to the origin has a negative sense. Hence vecotrs $\boldsymbol{d} \equiv (x, y, z)$ and $-\boldsymbol{d} \equiv (-x, -y, -z)$ represent the same line in space but their directions are of opposite senses. We define the length of a vector $\boldsymbol{d} \equiv (x, y, z)$ (sometimes called its modulus, or absolute value) as $| \boldsymbol{d} |$, and the distance of the point vector from the origin is

$$| \boldsymbol{d} | = \sqrt{}(x^2 + y^2 + z^2)$$

So any point on the line $\boldsymbol{p} + \mu\boldsymbol{d}$ is found by moving to the point $\boldsymbol{p}$ and then travelling along a line that is parallel to the direction $\boldsymbol{d}$, a distance of $\mu \mid d \mid$ in the positive sense of $\boldsymbol{d}$ if μ is positive, and in the negative sense otherwise. Note that any point on the line can act as a base vector, and the directional vector may be replaced by any non-zero scalar multiple of itself.

If the directional vector $\boldsymbol{d} \equiv (x, y, z)$ makes angles of θx, θy and θz with the respective positive x-direction, y-direction and z-direction, then

$$x : y : z = \cos\theta_x : \cos\theta_y : \cos\theta_z$$

which means that

$$\boldsymbol{d} \equiv (\lambda \times \cos\theta_x, \lambda \times \cos\theta_y, \lambda \times \cos\theta_z) \text{ for some } \lambda.$$

We know from the properties of three-dimensional geometry that

$$\cos^2\theta_x + \cos^2\theta_y + \cos^2\theta_z = 1$$

Hence $\lambda = \mid \boldsymbol{d} \mid$, and if the directional vector has unit modulus (that is, modulus = $\lambda = 1$), then the coordinates of this vector must be $\cos\theta_x$, $\cos\theta_y$, $\cos\theta_z$. The coordinates of a directional vector given in this way are called the direction cosines of the set of lines that is generated by the vector. In general, if the direction vector is $\boldsymbol{d} \equiv (x, y, z)$ then the direction cosines are

$$\left(\frac{x}{\mid \boldsymbol{d} \mid}, \frac{y}{\mid \boldsymbol{d} \mid}, \frac{z}{\mid \boldsymbol{d} \mid} \right)$$

Example 7.1

Describe the line joining (1 , 2, 3) to (–1 , 0, 2), by using the three methods shown so far

The general point (x, y, z) on the line satisfies the equations

$$(x - 1) \times (0 - 2) = (y - 2) \times (-1 - 1)$$

$$(y - 2) \times (2 - 3) = (z - 3) \times (0 - 2)$$

$$(z - 3) \times (-1 - 1) = (x - 1) \times (2 - 3)$$

That is

$$-2x + 2y = 2 \qquad (7.1)$$

$$-y + 2z = 4 \qquad (7.2)$$

$$-2x + x = -5 \qquad (7.3)$$

Note that equation (7.1) is –2 times the sum of equations (7.2) and (7.3). Thus we need consider only these latter two equations, to get

$$x = 2z - 5$$

$$y = 2x - 4$$

Hence the general point on the line depends only on one variable, in this case z, and it is giwn by $(2z - 5, 2z - 4, z)$. This result can easily be checked by noting that when $z = 3$ we get (1, 2, 3) and when $z = 2$ we get (−1, 0, 2), the two original points that define the line.

In vector form the general point on the line (depending on μ) is

$$\boldsymbol{p}(\mu) \equiv (1 - \mu)(1, 2, 3) + \mu(-1, 0, 2) \equiv (1 - 2\mu, 2 - 2\mu, 3 - \mu)$$

Again the coordinates depend on just one variable (π), and to check the validity of this representation of a line we note that $\boldsymbol{p}(0) \equiv (1, 2, 3)$ and $\boldsymbol{p}(1) \equiv (-1, 0, 2)$.

If we put the line into base/directional vector form we see that

$$\boldsymbol{p}(\mu) \equiv (1, 2, 3) + \mu(-2, -2, -1)$$

with (1, 2, 3) as the base vector and (−2, −2, −1) as the direction (which incidently has modulus $\sqrt{(4 + 4 + 1)} = \sqrt{9} = 3$). We also noted that any point on the line can act as a base vector, and so we can give another form for the general point on this line, $\boldsymbol{p}'$:

$$\boldsymbol{p}'(\mu) \equiv (-1, 0, 2) + \mu(-2, -2, -1)$$

We can change the directional vector into its direction cosine form (−2/3, −2/3, −1/3) and represent the line in another version of the base/direction form:

$$\boldsymbol{p}''(\mu) \equiv (1, 2, 3) + \mu(-2/3, -2/3, -1/3)$$

Naturally the same m value will give diff$rent points for different representations of the line; for example $\boldsymbol{p}(3) \equiv (-5, -4, 0)$, $\boldsymbol{p}'(3) \equiv (-7, -6, -1)$ and $\boldsymbol{p}''(3) \equiv (-1, 0, 2)$. The direction of this line makes angles of 131.81 degrees (= $\cos^{-1}(-2/3)$), 131.81 degrees and 109.47 degrees (= cos–1 (−1/3)) witln the positive x-direction, y-direction and z-direction respectively.

The Angle between Two Directional Vectors

In order to calculate such an angle we first introduce the operator ·, the dot product or scalar product. This operates on two vectors and returns a scalar (real) result thus:

$$\boldsymbol{p} \bullet \boldsymbol{q} = (x_1, y_1, z) \bullet (x_2, y_2, z_2) = x_1 \times x_2 + y_1 \times y_2 + z_1 \times z_2$$

If $\boldsymbol{p}$ and $\boldsymbol{q}$ are both unit vectors (that is, they are in direction cosine form), and $\boldsymbol{q}$ is the angle between the lines, then $\cos\theta = \boldsymbol{p} \bullet \boldsymbol{q}$ (see chapter 3 for the equivalent two-dimensional relationship). In general, therefore, the angle between two directional vectors p and q (we can assume they meet at the origin) is

$$\cos^{-1}\left(\frac{\boldsymbol{p}}{|\boldsymbol{p}|} \bullet \frac{\boldsymbol{q}}{|\boldsymbol{q}|}\right)$$

Obviously $\boldsymbol{p}$ and $\boldsymbol{q}$ are mutually perpendicular directions if and only if $\boldsymbol{p} \bullet \boldsymbol{q} = 0$.

Definition of Plane

The plane is the next object we must consider in three-dimensional space. The general point $x \equiv (x, y, z)$ on the plane is given by the vector equation:

$$\boldsymbol{n} \bullet \boldsymbol{x} = k$$

where k is a scalar, and $\boldsymbol{n}$ is the directional vector of the set of lines that are perpendicular to (or norma1 to) the plane (see example 7.2). If a is any point on the plane then naturally $\boldsymbol{n} \bullet \boldsymbol{a} = k$ and so by replacing k in the above equation, we may rewrite it as

$$\boldsymbol{n} \bullet \boldsymbol{x} = \boldsymbol{n} \bullet \boldsymbol{a} \quad \text{or} \quad \boldsymbol{n} \bullet (\boldsymbol{x} - \boldsymbol{a}) = 0$$

This latter equation is self-evident from the property of the dot product - two mutually perpendicular lines have zero dot product. For any point $x \equiv (x, y, z)$ in the plane that is not equal to $\boldsymbol{a}$, we know that $(\boldsymbol{x} - \boldsymbol{a})$ can be considered as the direction of a line in the plane. Since $\boldsymbol{n}$ is normal to the plane, and incidently perpendicular to every line in the plane, then $\boldsymbol{n} \bullet (\boldsymbol{x} - \boldsymbol{a}) = \cos(\pi/2) = 0$.

By expanding the original equation of the plane with normal $\boldsymbol{n} \equiv (n_1, n_2, n_3)$, we get the usual coordinate representation of a plane:

$$(n_1, n_2, n_3) \bullet (x, y, z) = n_1 \times x + n_2 \times y + n_3 \times z) = k$$

Note that two planes with normals $\boldsymbol{n}$ and $\boldsymbol{m}$ (say) are parallel if and only if one normal is a scalar multiple of the other, that is if $\boldsymbol{n} = \lambda\boldsymbol{m}$ for some $\lambda \neq 0$.

The Point of Intersection of a Line and a Plane

Suppose the line is given by $\boldsymbol{b} + \mu\boldsymbol{d}$ and the plane by $\boldsymbol{n} \bullet \boldsymbol{x} = k$. Since the point of intersection lies on both the line and the plane we have to find the unique value of μ (if one exists) for which

$$\boldsymbol{n} \bullet (\boldsymbol{b} + \mu\boldsymbol{d}) = k$$

that is

$$\mu = (k - \boldsymbol{n} \bullet \boldsymbol{b})/(\boldsymbol{n} \bullet \boldsymbol{d}) \text{ provided } \boldsymbol{n} \bullet \boldsymbol{d} \neq 0$$

$\boldsymbol{n} \bullet \boldsymbol{d} = 0$ if the line and plane are parallel and so either there is no point of intersection or the line is in the plane.

The Distance of a Point from a Plane

The distance of a point $\boldsymbol{p}_1$ from a plane $\boldsymbol{n} \bullet \boldsymbol{x} = k$ is the distance of $\boldsymbol{p}_1$ from the nearest point $\boldsymbol{p}_2$ on the plane. Hence the normal from the plane at $\boldsymbol{p}_2$ must pass through $\boldsymbol{p}_1$. This line can be written $\boldsymbol{p}_1 + \mu\boldsymbol{n}$, and the μ value that defines $\boldsymbol{p}_2$ is such that

$$\mu = (k - \boldsymbol{n} \bullet \boldsymbol{p}_1)/(\boldsymbol{n} \bullet \boldsymbol{n})$$

from the equation above, and the distance of the point $\boldsymbol{p}_2 \equiv \boldsymbol{p}_1 + \mu\boldsymbol{n}$ from $\boldsymbol{p}_1$ is

$$\mu \times |\, \boldsymbol{n} \,| = |\, k - \boldsymbol{n} \bullet \boldsymbol{p}_1 | \,/\, |\, \boldsymbol{n} \,|$$

In particular, if $\boldsymbol{p}_1$ is the origin 0 then thc distance of the plane from the origin is $|\, k \,| \,/\, |\, \boldsymbol{n} \,|$. Furthermore, if $\boldsymbol{n}$ is a direction cosine vector we see that the distance of the origin from the plane is $|\, k \,|$, the absolute value of the real number k.

Example 7.2

Find the point of intersection of the line joining (1, 2, 3) to (–1, 0, 2) with the plane $(0, -2, 1) \bullet x = 5$, and also find the distance of the plane from the origin.

$$\boldsymbol{n} \equiv (0, -2, 1)$$

$$\boldsymbol{b} \equiv (1, 2, 3)$$

$$\boldsymbol{d} \equiv (-1, 0, 2) - (1, 2, 3) \equiv (-2, -2, -1)$$

$$\boldsymbol{n} \equiv \boldsymbol{b} = (0 \times 1 + 2 \times 2 + 1 \times 3) = -1$$

$$\boldsymbol{n} \equiv \boldsymbol{d} = (0 \times -2 + -2 \times -2 + 1 \times -1) = 3$$

hence the m value of the point of intersection is $(5 - (-1))/3 = 2$, and the point vector is

$$(1, 2, 3) + 2(-2, -2, -1) \equiv (-3, -2, 1)$$

and the distance from the origin is $5/\,|\, \boldsymbol{n} \,| = 5/\sqrt{5} = \sqrt{5}$.

The program given in listing 7.1 enables us to calculate the point of intersection (array P) of a line and a plane. The line has base vector B and direction D, and the plane has normal N and plane constant K. Note that, since we are working with decimal numbers, and thus are subject to rounding errors, we cannot check if a dot product is zero. We can find only if it is sufficiently small to be considered zero, and what is meant by sufficiently small is left to the programmer (on the BBC micro about six places after the decimal point is reasonable).

The Point of Intersection of Two Lines

Suppose we have two lines $\boldsymbol{b}_1 + \mu\boldsymbol{d}_1$ and $\boldsymbol{b}_2 + \lambda\boldsymbol{d}_2$. Their point of intersection, if it exists (if the lines are not coplanar or are parallel then they will not intersect), is identified by finding unique values of m and l that satisfy the vector equation (three separate coordinate equations):

$$\boldsymbol{b}_1 + \mu\boldsymbol{d}_1 = \boldsymbol{b}_2 + \lambda\boldsymbol{d}_2$$

Listing 7.1

```
100 REM Intersection of line and plane
110 DIM B(3),D(3),N(3),P(3)
119 REM input line and plane data
120 CLS :PRINT TAB(0,2)," Intersection of line and plane          "
130 INPUT" Base vector of line ",B(1),B(2),B(3)
140 INPUT" Direction vector of line ",D(1),D(2),D(3)
150 INPUT" Normal to plane ",N(1),N(2),N(3)
160 INPUT" Plane constant ",K
170 DOT=N(1)*D(1)+N(2)*D(2)+N(3)*D(3)
179 REM output line and plane data
180 CLS
190 PRINT TAB(0,5);"Base vector of line "
200 PRINT TAB(0,6);"(";B(1);",";B(2);",";B(3);")"
210 PRINT TAB(0,8);"Direction vector of line "
220 PRINT TAB(0,9);"(";D(1);",";D(2);",";D(3);")"
230 PRINT TAB(0,11);"Normal to plane "
240 PRINT TAB(0,12);"(";N(1);",";N(2);",";N(3);")"
250 PRINT TAB(0,14);"Plane constant  ";K
260 PRINT TAB(0,18);"Point of intersection"
269 REM find point of intersection
270 IF ABS(DOT)<0.000001 THEN PRINT TAB(22,18) "does not exist" : GOTO 330
280 MU=(K-N(1)*B(1)-N(2)*B(2)-N(3)*B(3))/DOT
290 FOR I%=1 TO 3
300 P(I%)=B(I%)+MU*D(I%)
310 NEXT I%
320 PRINT TAB(0,19);"(";P(1);",";P(2);",";P(3);")"
330 PRINT TAB(0,22); : STOP
```

Three equations in two unknowns means that for the equations to be meaningful there must be at least one pair of the equations that are independent, and the remaining equation must be a combination of these two. Two lines are parallel if one directional vector is a scalar multiple of the other. So we take two independent equations, find the values of μ and λ (we have two equations in two unknowns), and put them in the third equation to see if they are consistent. Example 7.3 will demonstrate this method, and listing 7.2 is a way of implementing it on a computer. The first line has base and direction stored in arrays B and D, and the second line in C and E: the calculated point of intersection goes into array P.

Note that if the two independent equatioils are

$$a_{11} \times m + d_{12} \times \lambda = b_1$$

$$a_{21} \times m + a_{22} \times \lambda = b_2$$

then the determinant of this pair of equations, $\Delta = a_{11} \times a_{22} - a_{12} \times a_{21}$ will be non-zero (because the equations are not related), and we have the solutions:

$$\mu = (a_{22} \times b_1 - a_{12} \times b_2)/\Delta \quad \text{and} \quad \lambda = (a_{11} \times b_2 - a_{21} \times b_1)/\Delta$$

Listing 7.2

```
100 REM Intersection of two lines
110 DIM B(3),D(3),C(3),E(3),N(3),P(3)
120 CLS : PRINT TAB(1,2)"Intersection of two lines",SPC(9)
129 REM input data on two lines
130 INPUT" Base vector of first line ",B(1),B(2),B(3)
140 INPUT" Direction vector of first line ",D(1),D(2),D(3)
150 INPUT" Base vector of second line "C(1),C(2),C(3)
160 INPUT" Direction vector of second line ",E(1),E(2),E(3)
169 REM output data on two lines
170 CLS
180 PRINT TAB(0,5);"Base vector of first line "
190 PRINT TAB(0,6);"(";B(1);",";B(2);",";B(3);")"
200 PRINT TAB(0,8);"Direction vector of first line "
210 PRINT TAB(0,9);"(";D(1);",";D(2);",";D(3);")"
220 PRINT TAB(0,11);"Base vector of second line "
230 PRINT TAB(0,12);"(";C(1);",";C(2);",";C(3);")"
240 PRINT TAB(0,14);"Direction vector of second line "
250 PRINT TAB(0,15);"(";E(1);",";E(2);",";E(3);")"
260 PRINT TAB(0,18);"Point of intersection"
269 REM find independent equations
270 FOR I%=1 TO 3
280 J%=(I% MOD 3)+1
290 DELTA=E(I%)*D(J%)-E(J%)*D(I%)
300 IF ABS(DELTA)>0.000001 THEN GOTO 330
310 NEXT I%
319 REM find point of intersection
320 PRINT TAB(22,18) "does not exist" : GOTO 410
330 MU=(E(I%)*(C(J%)-B(J%))-E(J%)*(C(I%)-B(I%)))/DELTA
340 LAMBDA=(D(I%)*(C(J%)-B(J%))-D(J%)*(C(I%)-B(I%)))/DELTA
350 K%=(J% MOD 3)+1
360 IF ABS(B(K%)+MU*D(K%)-C(K%)-LAMBDA*E(K%)) > 0.000001 THEN GOTO 320
370 FOR I%=1 TO 3
380 P(I%)=B(I%)+MU*D(I%)
390 NEXT I%
400 PRINT TAB(0,19);"(";P(1);",";P(2);",";P(3);")"
410 PRINT TAB(0,22); : STOP
```

Example 7.3

Find the point of intersection (if any) of

(a) $(1, 1, 1) + \mu(2, 1, 3)$ with $(0, 0, 1) + \lambda(-1, 1, 1)$

(b) $(2, 3, 4) + \mu(1, 1, 1)$ with $(-2, -3, -4) + \lambda(1, 2, 3)$

In (a) the three equations are

$$1 + 2\mu = 0 - \lambda \tag{7.4}$$

$$1 + \mu = 0 + \lambda \tag{7.5}$$

$$1 + 3\mu = 1 + \lambda \tag{7.6}$$

From equations (7.4) and (7.5) we get $\mu = -2/3$ and $\lambda = 1/3$, which when substituted in equation (7.6) gives $1 + 3 \times (-2/3) = -1$ on the left-hand side and $1 + 1 \times (1/3) = 4/3$ on the right hand side, which are obviously unequal so the lines do not intersect.

From (b) we get the equations

$$2 + \mu = -2 + \lambda \tag{7.7}$$

$$3 + \mu = -3 + 2\lambda \tag{7.8}$$

$$4 + \mu = -4 + 3\lambda \tag{7.9}$$

and from equations (7.7) and (7.8) we get $\mu = -2$ and $\lambda = 2$, and these values also satisfy equation (7.9) (lefr-hand side = right-hand side = 2). So the point of intersection is

$$(2, 3, 4) - 2(1, 1, 1) = (-2, -3, -4) + 2(1, 2, 3) = (0, 1, 2)$$

The Plane through Three Non-collinear Points

In order to solve this problem we must introduce a new vector operator, X the *vector product*, which operates on two vectors p and q (say) giving the vector result

$$\boldsymbol{p} \times \boldsymbol{q} = (p_1, p_2, p_3) \times (q_1, q_2, q_3) = (p_2 \times q_3 - p_2 \times q_2, p_3 \times q_1 - p_1 \times q_3,$$
$$p_1 \times q_2 - p_2 \times q_1$$

If $\boldsymbol{p}$ and $\boldsymbol{q}$ are non-parallel directional vectors then $\boldsymbol{p} \times \boldsymbol{q}$ is the directional vector that is perpendicular to both $\boldsymbol{p}$ and $\boldsymbol{q}$. It should also be noted that this operation is non-commutative. That is, in general for given values of $\boldsymbol{p}$ and $\boldsymbol{q}$ we note that $\boldsymbol{p} \times \boldsymbol{q} \neq \boldsymbol{q} \times \boldsymbol{p}$; these two vector products will represent directions on the same line but with opposite sense. For example (1, 0, 0) X (0, 1, 0) = (0, 0, 1) but (0, 1, 0) X (1, 0, 0) = (0, 0, –1); (0, 0, 1) and (0, 0, –1) are both parallel to the *z*-axis (and so perpendicular to the directions (1, 0, 0) and (0, 1, 0)), but they are of opposite sense. Usting 7.3 gives a main program that calls the procedures 'vecprod' (for the vector product of two vectors L and M returning vector N) and 'dotprod' (which calculates the dot product DOT of the vectors L and M).

Suppose we are given three non-collinear points $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$. Then the two vectors $\boldsymbol{p}_2 - \boldsymbol{p}_1$ and $\boldsymbol{p}_3 - \boldsymbol{p}_1$ represent the directions of two lines that are coincident at $\boldsymbol{p}_1$, both of which lie in the plane that contains the three points. We know that the normal to the plane is perpendicular to every line in the plane, in particular to the two lines mentioned above. Also, because the points are not collinear, $\boldsymbol{p}_2 - \boldsymbol{p}_1 \neq \boldsymbol{p}_3 - \boldsymbol{p}_1$, the normal to the plane is $(\boldsymbol{p}_2 - \boldsymbol{p}_1) \times (\boldsymbol{p}_3 - \boldsymbol{p}_1)$, since $\boldsymbol{p}_1$ lies in the plane the equation is

$$((\boldsymbol{p}_2 - \boldsymbol{p}_1) \times (\boldsymbol{p}_3 - \boldsymbol{p}_1)) \bullet (x - \boldsymbol{p}_1) = 0$$

Example 7.4
Give the coordinate equation of the plane through the points (0, 1, 1), (1, 2, 3) and (–2, 3, –1).

Listing 7.3

```
100 REM Example of dot/vector product
110 DIM L(3),M(3),N(3) : CLS
120 PRINT TAB(0,3)"Example of dot/vector product",SPC(10)
130 INPUT" Vector L ",L(1),L(2),L(3)
140 INPUT" Vector M ",M(1),M(2),M(3)
150 CLS : PROCvecprod
160 PRINT TAB(0,5);"Vector L "
170 PRINT TAB(0,6);"(";L(1);",";L(2);",";L(3);")"
180 PRINT TAB(0,8);"Vector M "
190 PRINT TAB(0,9);"(";M(1);",";M(2);",";M(3);")"
200 PRINT TAB(0,11);"Vector Product "
210 PRINT TAB(0,12);"(";N(1);",";N(2);",";N(3);")"
220 PRINT TAB(0,14);"Dot Product  "; FNdotprod
230 PRINT TAB(0,22) : STOP

300 REM vecprod
310 DEF PROCvecprod
320 LOCAL I%,J%,K%
330 FOR I%=1 TO 3
340 J%=(I% MOD 3)+1 : K%=(J% MOD 3)+1
350 N(I%)=L(J%)*M(K%)-L(K%)*M(J%)
360 NEXT I%
370 ENDPROC

400 REM dotprod
410 DEF FNdotprod=L(1)*M(1)+L(2)*M(2)+L(3)*M(3)
```

This is given by the general point $x \equiv (x, y, z)$ where

$$(((1, 2, 3) - (0, 1, 1)) \times ((-2, 3, -1) - (0, 1, 1))) \bullet (x, y, z)$$
$$-(0, 1, 1) = 0$$

that is

$$((1, 1, 2) \times (-2, 2, -2)) \bullet (x, y - 1, z - 1) = 0$$

or

$$(-6, -2, 4) \bullet (x, y - 1, z - 1) = 0$$

which in coordinate form is $-6x - 2y + 4z - 2 = 0$ or in the equivalent form $3x + y - 2z = -1$

The Point of Intersection of Three Planes

We assume that the three planes are defined by equations (7.10) to (7.12) below. The point of intersection of these three planes, $\boldsymbol{b} \equiv (x, y, z)$, must be in all three planes and satisfy

$$\boldsymbol{n}_1 \bullet \boldsymbol{x} = k_1 \tag{7.10}$$

$$\boldsymbol{n}_2 \bullet \boldsymbol{x} = k_2 \tag{7.11}$$

$$\boldsymbol{n}_3 \bullet \boldsymbol{x} = k_3 \tag{7.12}$$

where $\boldsymbol{n}_1 \equiv (n_{11}, n_{12}, n_{13})$, $\boldsymbol{n}_2 \equiv (n_{21}. n_{22}, n_{23})$ and $\boldsymbol{n}_3 \equiv (n_{31}, n_{32}, n_{33})$. We can rewrite these three equations as one matrix equation

$$\begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

and so the solution for $\boldsymbol{b}$ is given by the *column vector*

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{pmatrix}^{-1} \times \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

So any calculation that requires the intersection of three planes necessarily involves the inversion of a 3 × 3 matrix. Listing 7.4 gives the Adjoint method of finding NINV, the inverse of matrix N. It also returns variable SNG which equals 0 if N is non-singular and 1 otherwise.

Listing 7.4

```
500 REM Find NINV, the inverse of 3x3 matrix N
        using the Adjoint method
510 DEF PROCinv
520 LOCAL I%,J%,NI%,NNI%
529 REM find DET, determinant of N
530 DET=0 : NI%=2 : NNI%=3
540 FOR I%=1 TO 3
550 DET=DET+N(1,I%)*(N(2,NI%)*N(3,NNI%)-N(3,NI%)*N(2,NNI%))
560 NI%=NNI% : NNI%= (NNI% MOD 3)+1
570 NEXT I%
579 REM if DET zero then N singular
580 IF ABS(DET)<0.000001 THEN SNG=1 : ENDPROC ELSE SNG=0
589 REM calculate NINV
590 NI%=2 : NNI%=3
600 FOR I%=1 TO 3
610 NJ%=2 : NNJ%=3
620 FOR J%=1 TO 3
630 NINV(J%,I%)=(N(NI%,NJ%)*N(NNI%,NNJ%)-N(NI%,NNJ%)*N(NNI%,NJ%))/DET
640 NJ%=NNJ% : NNJ%= (NNJ% MOD 3)+1
650 NEXT J%
660 NI%=NNI% : NNI%= (NNI% MOD 3)+1
670 NEXT I%
680 ENDPROC
```

Again in the program to solve this problem (listing 7.5), vectors are represented as one-dimensional arrays, thus array B will contain the solution of the equations (b); array K will contain the plane constants. We are given the normals $\boldsymbol{n}_1$, $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$ in the form of a 3 × 3 array N, so the values in B are found by the following code. Obviously if any two of the planes are parallel or the three meet in a line, then SNG equals 1 and there is no unique point of intersection.

Listing 7.5

```
100 REM Intersection of three planes
110 DIM N(3,3),NINV(3,3),K(3),B(3)
120 CLS :PRINT TAB(0,2),"Intersection of three planes",SPC(10)
129 REM input data on planes  put in arrays N and K
130 FOR I%=1 TO 3
140 PRINT"Input normal and constant for plane ";I%
150 INPUT N(I%,1),N(I%,2),N(I%,3),K(I%)
160 NEXT I%
169 REM output data on planes
170 CLS
180 PRINT TAB(2,5);"PLANE No.   CONSTANT    NORMAL"
190 ROW=7
200 FOR I%=1 TO 3
210 PRINT TAB(0,ROW),I%,K(I%),"      (";
     N(I%,1);",";N(I%,2);",";N(I%,3);")"
220 ROW=ROW+2
230 NEXTI%
239 REM find NINV, the inverse of N and B, point of intersection
240 PRINT TAB(0,14);"Point of intersection"
250 PROCinv
260 IF SNG THEN PRINT TAB(22,14) "does not exist" : GOTO 340
270 FOR I%=1 TO 3
280 B(I%)=0
290 FOR J%=1 TO 3
300 B(I%)=B(I%)+NINV(I%,J%)*K(J%)
310 NEXT J%
320 NEXT I%
330 PRINT TAB(0,15);"(";B(1);",";B(2);",";B(3);")"
340 PRINT TAB(0,22); : STOP
```

Example 7.5

Find the point of intersection of the three planes (0, 1, 1) • $\boldsymbol{x}$ = 2, (1,2, 3) • x = 4 and (1, 1, 1) • $\boldsymbol{x}$ = 0.

In the matrix furm we have

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$$

The inverse of $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ is $\begin{pmatrix} -1 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$

and so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$$

This solution is easily checked: (0, 1, 1) – (–2, 0, 2) = 2, (1, 2, 3) – (–2, 0, 2) = 4 and (1, 1, 1) – (–2, 0, 2) = 0, which means the point (–2, 0, 2) lies on all three planes and so is their point of intersection.

The Line of Intersection of Two Planes

Let the two planes be

$$\boldsymbol{p} \bullet \boldsymbol{x} = (p_1, p_2, p_3) \bullet \boldsymbol{x} = k_1$$

and

$$\boldsymbol{q} \bullet \boldsymbol{x} = (q_1, q_2, q_3) \bullet \boldsymbol{x} = k_2$$

We assume that the planes are not parallel, and so $\boldsymbol{p} \neq \boldsymbol{q}$ for all λ. The line common to the two planes naturally lies in each plane, and so it must be perpendicular to the normals of both planes (***p*** and ***q***). Thus the direction of this line must be $\boldsymbol{d} \equiv \boldsymbol{p} \times \boldsymbol{q}$ and the line can be written in the form $\boldsymbol{b} + \mu\boldsymbol{d}$ where ***b*** can be any point on the line. In order completely to classify the line we have to find one such b. We find a point that is the intersection of the two planes together with a third that is neither parallel to them nor cuts them in a common line. By choosing a plane with normal $\boldsymbol{p} \times \boldsymbol{q}$ we shall satisfy these conditions (and remember we have already calculated this vector product). We still need a value for k_3 but any value will do, so we take $k_3 = 0$ in order that this third plane goes through the origin. Thus b is given by the column vector

$$\boldsymbol{b} = \begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ p_2 \times q_3 - p_3 \times q_2 & p_3 \times q_1 - p_1 \times q_3 & p_1 \times q_2 - p_2 \times q_2 \end{pmatrix}^{-1} \times \begin{pmatrix} k_1 \\ k_2 \\ 0 \end{pmatrix}$$

Find the line that is cominon to the planes (0, 1, 1) • ***x*** = 2 and (1, 2, 3) • ***x*** = 2.

p = (0, 1, 1) and ***q*** = (1, 2, 3), and so $\boldsymbol{p} \times \boldsymbol{q}$ = (1 × 3 – 1 × 2, 1 x 1 – 0 × 3, 0 × 2 – 1 × 1) = (1, 1, – 1). We require the inverse of

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 1 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -5 & 2 & 1 \\ 4 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

and hence the point of intersection of the three planes is

$$\frac{1}{3}\begin{pmatrix} -5 & 2 & 1 \\ 4 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -6 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$$

and the line is $(-2, 2, 0) + \mu(1, 1, -1)$.

It is easy to check this result, because all the points on the line should lie both planes:

$$(0, 1, 1) \bullet ((-2, 2, 0) + \mu(1, 1, -1))$$

$$= (0, 1, 1) \bullet (-2, 2, 0) + \mu(0, 1, 1) \bullet (1, 1, -1) = 2 \text{ for all } \mu$$

and

$$(1, 2, 3) \bullet ((-2, 2, 0) + \mu(1, 1, -1))$$

$$= (0, 1, 1) \bullet (-2, 2, 0) + \mu(1, 2, 3) \bullet (1, 1, -1) = 2 \text{ for all } \mu$$

The program to solve this problem is given as listing 7.6; note that it is very similar to the previous program. Also note that arrays are not explicitly used for $\boldsymbol{p}$ and $\boldsymbol{q}$ - these values are stored in the first two rows of array N. Array B holds the base vector of the line of intersection, but we do not place d in an array because the values are already in the third row of N.

Functional representation of a Surface

In our study of two-dimensional space in chapter 3 we noted that curves can b represented in a functional notation. This idea can be extended into three dimensions when we study surfaces. The simplest form of surface is an infinite plane with normal $\boldsymbol{n} \equiv (n_1, n_2, n_3)$, which we have seen can be given as coordinate equation:

$$\boldsymbol{n} \bullet \boldsymbol{x} - k = n_1 \times x + n_2 \times y_2 + n_3 \times z - k = 0$$

This can be written in functional form for a general point $x \equiv (x, y, z)$ on the surface:

$$f(x) \equiv f(x, y, z) \equiv n_1 \times x + n_2 \times y + n_3 \times z - k \equiv \boldsymbol{n} \bullet \boldsymbol{x} - k$$

which is a simple expression in variables x, y and $z(\boldsymbol{x})$. This enables us to divide all the points in space into three sets, those with $f(\boldsymbol{x}) = 0$ (the zero set), those with $f(\boldsymbol{x}) < 0$ (the negative set) and those with $f(\boldsymbol{x}) > 0$ (the positive set). A point $\boldsymbol{x}$ lies on the surface if and only if it belongs to the zero set. If the surface divides space into two halves (each half being *connected*, that is any two points in a given half can be joined by a curve that does not cross the surface) then these two halves may be identified with the positive and negative sets. Again beware, there are

Listing 7.6

```
100 REM Intersection of two planes
110 DIM N(3,3),NINV(3,3),K(3),B(3)
120 CLS : PRINT TAB(0,2),"Intersection of two planes",SPC(10)
129 REM input plane information
130 FOR I%=1 TO 2
140 PRINT"Input normal and constant for plane ";I%
150 INPUT N(I%,1),N(I%,2),N(I%,3),K(I%)
160 NEXT I%
169REM find third rows of N and K directional vector of the
       line of intersection is (N(3,1),N(3,2),N(3,3))
170 N(3,1)=N(1,2)*N(2,3)-N(1,3)*N(2,2)
180 N(3,2)=N(1,3)*N(2,1)-N(1,1)*N(2,3)
190 N(3,3)=N(1,1)*N(2,2)-N(1,2)*N(2,1)
200 K(3)=0
209 REM output plane information
210 CLS
220 PRINT TAB(2,5);"PLANE No.   CONSTANT    NORMAL"
230 ROW=7
240 FOR I%=1 TO 2
250 PRINT TAB(0,ROW),I%,K(I%),"
(";N(I%,1);",";N(I%,2);",";N(I%,3);")"
260 ROW=ROW+2
270 NEXTI%
279 REM compare with listing 7.5
280 PRINT TAB(0,13);"Line of intersection"
290 PROCinv
300 IF SNG THEN PRINT TAB(22,13) "does not exist" : GOTO 410
310 FOR I%=1 TO 3
320 B(I%)=0
330 FOR J%=1 TO 3
340 B(I%)=B(I%)+NINV(I%,J%)*K(J%)
350 NEXT J%
360 NEXT I%
369 REM output line of intersection
370 PRINT TAB(0,15);"Base vector"
380 PRINT TAB(0,16);"(";B(1);",";B(2);",";B(3);")"
390 PRINT TAB(0,18);"Directional vector"
400 PRINT TAB(0,19);"(";N(3,1);",";N(3,2);",";N(3,3);")"
410 PRINT TAB(0,22); : STOP
```

many surfaces that divide space into more than two connected volumes and then it is impossible to relate functional representation with connected sets; for example $f(x, y, z) \equiv \cos(y) - \sin(x^2 + z^2)$. There are, however, many useful well-behaved surfaces with this property, the sphere of radius r for example:

$$f(x) \equiv r^2 - |\, x \,|^2$$

that is

$$f(x, y, z) \equiv r^2 - x^2 - y^2 - z^2$$

If $f(x) = 0$ then x lies on the sphere, If $f(x) < 0$ then x lies outside the sphere, and if $f(x) > 0$ then x lies inside the sphere.

The functional representation of a suface is a very useful concept. ut can be used to define sets of equations that are necessary in calculating the intersections of various objects. The major use, however, is to determine whether or not two points $\boldsymbol{p}$ and $\boldsymbol{q}$ (say) lie on the same side of a surface that divides space into two parts. All we need to do is compare the signs of $f(\boldsymbol{p})$ and $f(\boldsymbol{q})$. If they are of opposite signs then a line joining $\boldsymbol{p}$ and $\boldsymbol{q}$ must cut the surface. Some examples are now given.

Is a point on the same side of a plane as the origin?
Suppose the plane is defined (as earlier) by three non-collinear points $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ and $\boldsymbol{p}_3$. Then the equation of the plane is

$$((\boldsymbol{p}_2 - \boldsymbol{p}_1) \times (\boldsymbol{p}_3 - \boldsymbol{p}_1)) \bullet (\boldsymbol{x} - \boldsymbol{p}_1) = 0$$

We may rewrite this in functional form

$$f(\boldsymbol{x}) \equiv ((\boldsymbol{p}_2 - \boldsymbol{p}_1) \times (\boldsymbol{p}_3 - \boldsymbol{p}_1)) \bullet (x - \boldsymbol{p}_1)$$

So all we need do for a point $\boldsymbol{e}$ (say) is to compare $f(\boldsymbol{e})$ with $f(\boldsymbol{O})$, where $\boldsymbol{O}$ is the origin. We assume here that neither $\boldsymbol{O}$ nor $\boldsymbol{e}$ lie in the plane.

We shall see that this idea will be of great use in the study of hidden surface algorithms.

Example 7.7
Are the origin and point (1, 1, 3) on the same side of the plane defined by points (0, 1, 1), (1, 2, 3) and (–2, 3, –1)?

From example 7.4 we see that the functional representation of the plane is

$$f(x) \equiv (-6, -2, 4) \bullet (\boldsymbol{x} - (0, 1, 1))$$

Thus

$$f(0, 0, 0) = -(-6, -2, 4) \bullet (0, 1, 1) = -2$$

and

$$f(1, 1, 3) = (-6, -2, 4) \bullet ((1, 1, 3) - (0, 1, 1)) = 2$$

Hence (1, 1, 3) lies on the opposite side of the plane to the origin and so a line segment that joints the two points will cut the plane at a point $(1 - \mu)$ (0, 0, 0) + $\mu(1, 1, 3)$ where $0 < \mu < 1$.

Is an oriented convex polygon of vertices in two dimensional space clockwise or anticlockwise?
We start by assuming that the polygon is a triangle that is defined by the three vertices $\boldsymbol{p}_1 \equiv (x_1, y_1)$, $\boldsymbol{p}_2 \equiv (x_2, y_2)$ and $\boldsymbol{p}_3 \equiv (x_3, y_3)$. Although these points are in two-dimensional space we can assume they lie in the x/y plane through the origin of three-dimensional space by giving them all a z-coordinate value of zero. We systematically define the directions of the edges of the polygon to be $(\boldsymbol{p}_2 - \boldsymbol{p}_1)$,

($\boldsymbol{p}_3 - \boldsymbol{p}_2$) and ($\boldsymbol{p}_1 - \boldsymbol{p}_2$). Since these lines all lie in the *x*/*y* plane through the origin we know that for all i = 1, 2 or 3 and for some real numbers ri that depend on i

$$(\boldsymbol{p}_{i+1} - \boldsymbol{p}_i) \times (\boldsymbol{p}_i + 2 - \boldsymbol{p}_{i+1}) = (0, 0, r_i)$$

This is because this vector product is perpendicular to the *x*/*y* plane and so only *z*-coordinate values may be non-zero. The addition of subscripts is modulo 3. Because the vertices were taken systematically, note that the signs of these r_i values are always the same; but what is more important, if the $\boldsymbol{p}_i$ values are clockwise then the r_i values are all negative, and if the $\boldsymbol{p}_i$ values are anticlockwise the r_i values are all positive.

Given an oriented convex polygon we need only consider the first three vertices to find if it is clockwise or anticlockwise. This technique will prove to be invaluable when we deal with hidden line/surface algorithms later in this book. Listing 7.7 allows us to find whether or not three ordered two-dimensional vertices form an anticlockwise triangle.

Listing 7.7

```
100 REM Orientation of 2-D triangle
110 DIM X(3),Y(3)
119 REM input data on triangle
120 CLS : PRINT TAB(0,3)"TRIANGLE DEFINED BY VERTICES"
130 ROW=2
140 FOR I%=1 TO 3
150 PRINT TAB(0,20) "Type in coordinates of vertex ";I%
160 INPUT X(I%),Y(I%)
170 PRINT TAB(0,21),SPC(32)
180 ROW=ROW+3
189 REM output data on triangle
190 PRINT TAB(0,ROW) "VERTEX ";I%
200 PRINT TAB(0,ROW+1);"(";X(I%);",";Y(I%);")"
210 NEXT I%
219 REM form two directional vectors (DX1,DY1,0) and (DX2,DY2,0)
220 DX1=X(2)-X(1) : DY1=Y(2)-Y(1)
230 DX2=X(3)-X(2) : DY2=Y(3)-Y(2)
240 PRINT TAB(0,15);"IS ";
249 REM check sign of z-coordinate of the vector product
250 IF DX1*DY2-DX2*DY1>0 THEN PRINT "ANTI-";
260 PRINT "CLOCKWISE"
270 PRINT TAB(0,20),SPC(32) : STOP
```

Example 7.8

Why is the polygon given in example 3.4 anticlockwise?

The vertices (considered in three dimensions) are (1, 0, 0), (5, 2, 0), (4, 4, 0) and (–2, 1, 0). The directions of the edges are (4, 2, 0), (–1, 2, 0), (–6, –3, 0) and (3, –1 , 0).

$$(4, 2, 0) \times (-1, 2, 0) = (0, 0, 10)$$

$$(-1, 2, 0) \times (-6, -3, 0) = (0, 0, 15)$$

$(-6, -3, 0) \times (3, -1, 0) = (0, 0, 15)$

$(3, -1, 0) \times (4, 2, 0) = (0, 0, 10)$

Since these are all positive, the orientation of the polygon is anticlockwise. But be careful, if you 1ose this consistent order for calculating the vector product you can get the wrong answer. For example

$(-6, -3, 0) \times (4, 2, 0) = (0, 0, 0)$ - the lines are parallel!

or

$(-1, 2, 0) \times (3, -1, 0) = (0, 0, -5)$ - the edges have been taken out of sequence.

Complete Programs

I Listing 7.1 (intersection of line and plane). Data required: a base vector (B(1), B(2), B(3)) and direction vector (D(1), D(2), D(3)) for the line, a normal (N(1), N(2), N(3)) and constant K for the plane. Try (1, 2, 3), (0, 2, −1), (1, 0, 1) and 2 respectively.

II Listing 7.2 (intersection oftwo lines). Data required: a base and direction vectors for the two lines, (B(1), B(2), B(3)) and (D(1), D(2), D(3)), and (C(1), C(2), C(3)) and (E(1), E(2), E(3)). Try (1, 2, 3), (1, 1, −1), and (−1, 1, 3), (1, 0, 1).

III Listing 7.3 ('main program', 'vecprod' and 'dotprod'). Data required: two vectors (L(1), L(2), L(3)) and (M(1), M(2), M(3)). Try (1, 2, 3), (1, 1 , −1).

IV Listings 7.4 ('inv') and 7.5 (intersection of three planes). Data required: normal (N(1, 1), N(1, 2), N(1, 3)) and constant K(1) for the three planes, $1 \le I \le 3$. Try (1, 2, 3), 0, (1, 1, −1), 1, (1, 0, 1), 2.

V Listings 7.4 ('inv') and 7.6 (intersection of two planes). Data required: normal (N(1, 1), N(1, 2), N(1, 3)) and constant K(1) for the two planes, $1 \le I \le 2$. Try (1, 2, 3), 0, (1, 1, − 1), 1.

VI Listing 7.7 (orientation of two-dimensional triangle). Data required: the vertices (X(1), Y(1)), $1 \le I \le 3$. Try (1, 2), (2, 3) and (−1, 1).