3 Two-Dimensional Coordinate Geometry

In chapter 2 we introduced the concept ofthe two-dimensional rectangular coordinate system, and defined points in space as vectors, whence we were able to draw line segments between pairs of points. To be strictly accurate, a *straight line* (or *line* for short) in two-dimensional space is not a finite segment, but stretches off to infinity in both directions, and so we need to introduce ways of representing a general point on such a line.

It is well known that the equation of a straight line is $y = mx + c$. This gives the relationship between the x-coordinate and the y-coordinate of a general point on a line, where m is the tangent of the angle that the line makes with the positive x-axis, and c is the point of intersection of the line with the y-axis, so that when $x = 0$ then $y = c$. This formula may be well known, but it is not very useful: if the line is vertical, then m is infinite! A far better formula is

$$ay = bx + c$$

This allows for all possible lines: if the line is vertical a is 0. (b/a) is now the tangent of the angle that the line makes with the positive x-axis, and the line cuts the y-axis at (c/a) provided that a is not equal to zero, and the x-axis at $(-c/b)$ provided that b is not equal to zero. The line is parallel to the y-axis if a is zero, and to the x–axis if b is zero.

We shall frequently use this fonnulation of a line in the following pages; however we now introduce another, possibly more useful, method for defining a line. Before we can describe this new method we must first define two operations on vectors (namely scalar multiple and vector addition) as well as describe how to calculate the absolute value of a vector. Suppose that we have two vectors, $\boldsymbol{p}_1 \equiv (x_1, y_1)$ and $\boldsymbol{p}_2 \equiv (x_2, y_2)$ then

scalar multiple: we multiply the individual coordinates by a scalar (real) value:

$$k\boldsymbol{p}_1 = (k \times \mathrm{i}_1, k \times y_1)$$

vector addition: we add the x-coordinates together, and the y-coordinates together.

$$\boldsymbol{p}_1 + \boldsymbol{p}_2 = (x_1 + x_2, y_1 + y_2)$$

absolute value: the distance of the point p2 from the origin (this is also called the length, and the modulus of the vector).

$| \boldsymbol{p}_1 | = \sqrt{(x_1^2 + y_1^2)}$

To define a line we first arbitrarily choose any two points on the line - again we call them $\boldsymbol{p}_1 \equiv (x_1, y_1)$ and $\boldsymbol{p}_2 \equiv (x_2, y_2)$. A general point $\boldsymbol{p}(\mu) \equiv (x, y)$ is given by the combination of scalar multiples and vector addition

$(1 - \mu)\boldsymbol{p}_1 + \mu\boldsymbol{p}_2$

for some reitl value of μ; that is the vector $((1 - \mu) \times x_1 + \mu\, x_2 , (1 - \mu) \times y_1 + \mu \times y_2)$. We place the μ in brackets after $\boldsymbol{p}$ to show the dependence of the vector on the value of μ. Later when we understand the relationship more fully we shall leave out the (μ). If $0 \le u \le 1$ then $p(\underline{\mu})$ lies on the line somewhere between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. For any specified point $\boldsymbol{p}(\mu)$, the value of μ is given by the ratio

$$\frac{\text{distance of } \boldsymbol{p}(\mu) \text{ from } \boldsymbol{p}_1}{\text{distance of } \boldsymbol{p}_2 \text{ from } \boldsymbol{p}_1}$$

where the measure of distance is positive if p(u) is on the same side of $\boldsymbol{p}_1$ as $\boldsymbol{p}_2$, and negative otherwise. The positive distance between any two vector points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ is given by (Pythagoras)

$| \boldsymbol{p}_2 - \boldsymbol{p}_1 | = \sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}}$

Figure 2.1 shows a line segment between the points $(-3, -1) \equiv \boldsymbol{p}(0)$ and $(3, 2) \equiv \boldsymbol{p}(1)$: the point (1 , 1) lies on the line as $\boldsymbol{p}(2/3)$. Note that (3, 2) is at a distance of $3\sqrt{5}$ from $(-3, -1)$ whereas (1 , 1) is at a distance of $2\sqrt{5}$. From now on we shall omit the (μ) from the point vector.

Example 3.1

We can further illustrate this idea by drawing the pattern shown in figure 3.1 . At first sight it looks complicated, but on closer inspection it is seen to be simply a square, outside a square, outside a square etc. The squares are getting successively sinaller and they are rotating through a constant angle. In order to draw the diagram we need a technique that, when given a general square, draws a smaller internal square rotated through this fixed angle. Suppose the general square has four corners $\{(x_i , y_i) \mid i = 1, 2, 3, 4\}$ and the i^{th} side of the square is the line joining (x_1, y_1) to (x_{i+1}, y_{i+1}) – assuming that additions of subscripts are modulo 4 (that is $4 + 1 \equiv 1$). A general point on this side of the square, (x'_i, y'_i), is given by

$((1 - \mu) \times x_i + \mu \times x_{i+1}, (1 - \mu) \times\ y_1, + \mu \times y_{i+1})$ where $0 \le \mu \le 1$

In fact $\mu{:}1 - \mu$ is the ratio in which the side is bisected. If μ is fixed and the four points $\{(x'_i , y'_i) \mid i = 1, 2, 3, 4 \}$ are calculated in the above manner, then the sides of the new square make an angle $\alpha = \tan^{-1} [\, \mu/(1 - \mu) \,]$ with the corresponding side of the outer square. So by keeping μ fixed for each new square, the angle between consecutive squares remains a constant α. In listing 3.1 , which generated figure 3.1, there are 21 squares and $\mu = 0.1$.

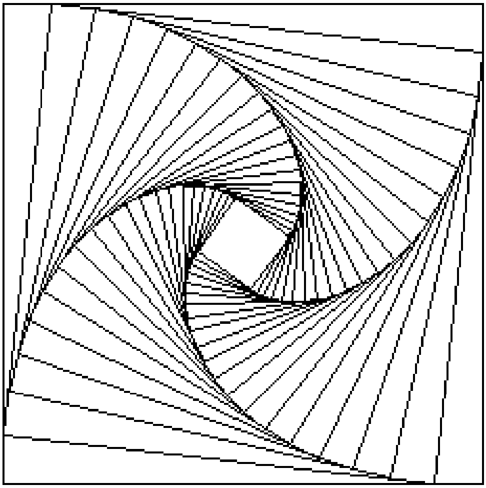

Figure 3.1

It is useful to note that the vector combination form of a line can be reorganised as

$$\boldsymbol{p}_1 + \mu(\boldsymbol{p}_2 - \boldsymbol{p}_1)$$

When given in this new representation the vector $\boldsymbol{p}_1$ may be called the base vector, and $(\boldsymbol{p}_2 - \boldsymbol{p}_1)$ the *directional vector*. In fact any point on the line can stand as a base vector; it simply acts as a point to anchor a line that is parallel to the directional vector. This concept of a vector acting as a direction needs some further explanation. We have already seen that a vector pair, (x, y) say, may represent a point; a line that joins the coordinate origin to this point may be thought of as specifying a direction - any line in space that is parallel to this line is defined to have the same directional vector. We insist that the line goes from the origin towards (x, y), the so-called positive sense; a line from (x, y) towards the origin has negative sense.

This base and direction representation is also very useful fur calculating the point of intersection of two lines, a problem that frequently crops up in two dimensional graphics. For suppose we have two lines $\boldsymbol{p} + \mu\boldsymbol{q}$, and $\boldsymbol{r} + \lambda\boldsymbol{s}$, where

Listing 3.1

```
100 REM square in square etc.
110 MODE 1
120 HORIZ=2.8 : VERT=2.1
130 PROCstart(3,0)
140 PROCsetorigin(HORIZ/2,VERT/2)
150 DIM X(4),Y(4),XD(4),YD(4)
160 DATA 1,1, 1,-1, -1,-1, -1,1
169 REM setup coordinates of square
170 FOR I%=1 TO 4 : READ X(I%),Y(I%) : NEXT I%
180 MU=0.1 : UM=1-MU
189 REM loop through 21 squares
190 FOR I%=1 TO 21
200 PROCmoveto(X(4),Y(4))
209 REM draw square defined by arrays X and Y
         : find next square given by arrays XD and YD
210 FOR J%=1 TO 4
220 PROClineto(X(J%),Y(J%))
230 NJ%=(J% MOD 4)+1
240 XD(J%)=UM*X(J%)+MU*X(NJ%)
250 YD(J%)=UM*Y(J%)+MU*Y(NJ%)
260 NEXT J%
269 REM reset (X,Y) values to (XD,YD)
270 FOR J%=1 TO 4
280 X(J%)=XD(J%) : Y(J%)=YD(J%)
290 NEXT J%
300 NEXT I%
310 STOP
```

$\boldsymbol{p} \equiv (x_1, y_1)$, $\boldsymbol{q} \equiv (x_2, y_2)$, $\boldsymbol{r} \equiv (x_3, y_3)$ and $\boldsymbol{s} \equiv (x_4, y_4)$ for $-\infty < \mu, \lambda < \mu$. We ned to find the unique values of μ and λ such that

$$\boldsymbol{p} + \mu\boldsymbol{q} = \boldsymbol{r} + \lambda\boldsymbol{s}$$

that is, a point that is common to both lines. This vector equation can be written as two separate equations

$$x_1 + \mu \times x_2 = x_3 + \lambda \times x_4 \tag{3.1}$$

$$y_1 + \mu \times y_2 = y_3 + \lambda \times y_4 \tag{3.2}$$

Rewriting these equations we get

$$\mu \times x_2 - \lambda \times x_4 = x_3 - x_1 \tag{3.3}$$

$$\mu \times y_2 - \lambda \times y_4 = y_3 - y_1 \tag{3.4}$$

Multiplying equation (3.3) by y_4, equation (3.4) by x_4 and subtracting we get

$$\mu \times (x_2 \times y_4 - y_2 \times x_4) = (x_3 - x_1) \times y_4 - (y_3 - y_1) \times x_4$$

If $(x_2 \times y_4 - y_2 \times x_4) = 0$ then the lines are parallel and there is no point of intersection (u does not exist), otherwise

$$\mu = \frac{(x_3 - x_1) \times y_4 - (y_3 - y_1) \times x_4}{(x_2 \times y_4 - y_2 \times x_4)} \tag{3.5}$$

and similary

$$\lambda = \frac{(x_3 - x_1) \times y_2 - (y_3 - y_1) \times x_2}{(x_2 \times y_4 - y_2 \times x_4)}$$

The solution becomes even simpler if one of the lines is parallel to a coordinate axis. Suppose this line is $x = d$, then we can set $\boldsymbol{r} \equiv (d, 0)$ and $\boldsymbol{s} \equiv (0, 1)$, which when substituted in equation (3.5) gives

$$\mu = (d - x_1)/x_2$$

and similarly if the line $y = d$

$$\mu = (d - y_1)/y_2$$

Naturally if the two lines are parallel then the denominator in these equations becomes zero and we get an infinite result, because two parallel lines cannot intersect.

Example 3.2

Find the point ofintersection ofthe two lines that (a) join (1 , –1) to (–1, –3) and (b) join (1, 2) to (3, –2).

The lines may be written as

$$(1 - \mu)(1, -1) + \mu(-1, -3) \qquad -\infty < \mu < \infty \qquad (3.7)$$

$$(1 - \lambda)(1, 2) + \lambda(3, -2) \qquad -\infty < \lambda < \infty \qquad (3,8)$$

or when placed in the base/directional form as

$$(1, -1) + \mu(-2, -2) \qquad (3.9)$$

$$(1, 2) + \lambda(2, -4) \qquad (3.10)$$

Substituting these values into equation (3.5) gives

$$\mu = \frac{(1 - 1) \times -4 - (2 + 1) \times 2}{(-2 \times 4 - (-2) \times 2)} = -1/2$$

whence the point of interaction is (1, –1) – 1/2(–2, –2) ≡ (2,0).

The general case is solved by the program given in listing 3.2.

Exercise 3.1

Experiment with this concept of vector representation of two-diimensional spac' . You can make up your own questions: it is easy to check that your answers are correct. Consider example 3.2. We know that (2, 0) lies on the first line because we used the value $\mu = -1/2$: our answer is correct if it also lies on the second line which it does with $\lambda = 1/2$.

Listing 3.2

```
100 REM intersection of two lines
110 MODE 7
120 DIM X(4),Y(4)
130 PRINT TAB(8,3),"INTERSECTION OF LINES"
140 PRINT TAB(0,5),"LINE A FROM (X(1),Y(1)) TO (X(2),Y(2))"
150 PRINT TAB(0,6),"LINE B FROM (X(3),Y(3)) TO (X(4),Y(4))  "
159 REM INPUT vertices of lines A & B
160 FOR I%=1 TO 4
170 PRINT "X(";I%;"),Y(";I%;") "; : INPUT X(I%),Y(I%)
180 NEXT I%
190 CLS
199 REM PRINT information about lines
200 PRINT TAB(0,5);"Line A goes from"
210 PRINT"(";X(1);",";Y(1);") to (";X(2);",";Y(2);")"
220 PRINT TAB(0,8);"Line B goes from"
230 PRINT"(";X(3);",";Y(3);") to (";X(4);",";Y(4);")"
239 REM calculate (XINT,YINT) the point of intersection
240 X(2)=X(2)-X(1) : Y(2)=Y(2)-Y(1)
250 X(4)=X(4)-X(3) : Y(4)=Y(4)-Y(3)
260 DET=X(2)*Y(4)-Y(2)*X(4)
270 PRINT TAB(0,12);"Point of intersection ";
280 IF ABS(DET) < 0.00001 THEN PRINT "does not exist." : GOTO 320
290 MU=((X(3)-X(1))*Y(4)-(Y(3)-Y(1))*X(4))/DET
300 XINT=X(1)+MU*X(2) : YINT=Y(1)+MU*Y(2)
310 PRINT : PRINT"(";XINT;",";YINT;")."
320 PRINT TAB(0,22); : STOP
```

Exercise 3.2

Write a program that reads in data about two straight lines in the form of an equation and then calculates their point of intersection (if any).

Returning to the use of a vector ($\boldsymbol{q} \equiv (x, y) \neq (0, 0)$, say) that represents a direction, we note that any positive scalar multiple $k\boldsymbol{q}$, for $k > 0$, represents the same direction and sense as q (if k is negative then the direction has its sense inverted). In particular, setting $k = 1/ \mid q \mid\}$ produces a vector $(x/\sqrt{(x^2 + y^2)}, y/\sqrt{(x^2 + y_2)})$ with unit absolute value.

Thus a general point on a line $\boldsymbol{p} + \mu q$, is a distance $\mu \mid \boldsymbol{q} \mid$ from the base point $\boldsymbol{p}$ and if $\mid \boldsymbol{q} \mid = 1$ (a unit vector) then the point is a distance μ from $\boldsymbol{p}$.

We now consider the angles made by directional vectors with various fixed diretions. Suppose that α is the angle between the line joining $\boldsymbol{O}$ (the origin) to $\boldsymbol{q} \equiv (x, y)$, and the positive x-axis. Then $x = \mid \boldsymbol{q} \mid \times \cos \alpha$ and $y = \mid \boldsymbol{q} \mid \times \sin \boldsymbol{a}$ - see figure 3.2: there are similar figures for the other three quadrents. If $\boldsymbol{q}$ is a unit vector (that is, $\mid \boldsymbol{q} \mid = 1$) then $\boldsymbol{q} \equiv (\cos \alpha, \sin \alpha)$. However, since $\sin \alpha = \cos(\alpha - \pi/2)$ for all values of α, this expression can be written as $\boldsymbol{q} \equiv (\cos \alpha, \cos(\alpha - \pi/2))$, where $\alpha - \pi/2$ is the angle that the vector makes with the positive y axis. Hence the coordinates of a unit directional vector are called its *direction cosines*, since they are the cosines of the angle that the vector makes with the corresponding positive axes

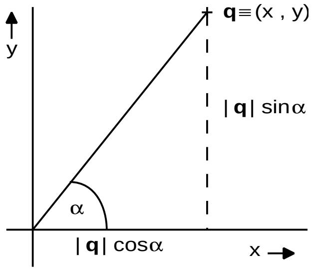

Figure 3.2

Before continuing we should take a look at the trigonometric functions available in BASIC: SIN and COS, and the inverse function ATN. SIN and COS are functions with one parameter (an angle given in radians) and one result (a value between –1 and +1). The ATN function takes any value and calculates the angle in radians (in the so-called *principal range* between $-\pi/2$ and $+\pi/2$) whose tangent is that value.

This leads us to the problem of finding the angle that a general direction q ≡ (x, y) makes with the positive x-axis, which is solved by the procedure ' angle' given in listing 3.3 . ‘angle’ will be of great use in later chapters when we consider three-dimensional space.

Listing 3.3

```
8810 DEF FNangle(AX,AY)
8820 IF ABS(AX)>0.00001 THEN 8860
8830 IF ABS(AY)<0.00001 THEN =0
8840 IF AY<0 THEN =1.5*PI
8850 =PI/2
8860 IF AX<0 THEN =(ATN(AY/AX)+PI) ELSE =ATN(AY/AX)
```

Now suppose we have two directional vectors (a, b) and (c, d); for simplicity we can assume that they are both unit vectors and that they both pass through the origin (see figure 3.3). We wish to calculate the acute angle, α, between these lines. From the figure we note that OA = $\sqrt{(a^2 + b_2)} = 1$ and OB = $\sqrt{(c_2 + d^2)} = 1$.

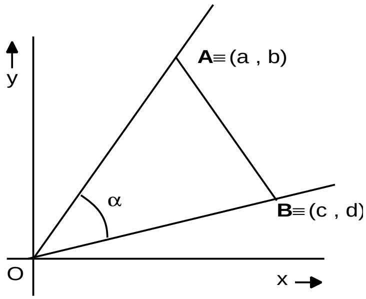

Figure 3.3

So by the Cosine Rule

$AB^2 = OA^2 + OB^2 - 2 \times OA \times OB \times \cos\alpha = 2 \times (1 - \cos\alpha)$

But also by Pythagoras

$AB^2 \quad = (a - c)2 + (b - d)^2 = (a^2 + b^2) + (c^2 + d^2) - 2 \times (a \times c + b \times d)$

$\quad = 2 - 2 \times (a \times c + b \times d)$

Thus $a \times c + b \times d = \cos\alpha$. It is possible that $a \times c + b \times d$ is negative in which case $\cos^{-1}(a \times c + b \times d)$ is obtuse and the required acute angle is $\pi - \alpha$. Since $\cos(\pi - \alpha) = -\cos\alpha$, then the acute angle is given immediately by $\cos^{-1}(| a \times c + b \times d |)$. For example, given the two lines with direction cosines $(\sqrt{3}/2, 1/2)$ and $(-1/2, -\sqrt{3}/2)$, we see that $a \times c + b \times c = -\sqrt{3}/2$ and thus $a = \cos^{-1}(\sqrt{3}/2) = \pi/6$. This simple example was given in order to introduce the concept of a *scalar product* • of two vectors, (a, b) • (c, d) = a × c + b × d. Scalar product is extendable into higher-dimensional space (see chapter 7 for a three-dimensional example) and it always has the property that it gives the cosine of the angle between any pair of lines whose directions are defined by the two vectors.

Curves: Functional Representations versus Parametric Forms

A curve in two-dimensional space can be considered as a relationship between *x* and *y* coordinate values, the so-called *functional relationship*. Alternatively the coordinates can be individually specified in terms of other variables or parameters, the *parametric form*.

We have already seen that a line (a circular arc of infinite radius) may be expressed as $ay = bx + c$. If we rearrange the equation so that one side is zero, that is $ay - bx = 0$, then the algebraic expression on the left-hand side of the equation is called a functional representation of the line and written as

$$f(x, y) \equiv ay - bx - c$$

All, but only, those points with the property $f(x, y) = 0$ lie on the curve. This representation divides all the points in two-dimensional space into three sets: $f(x, y) = 0$ (the zero set) $f(x,y) > 0$ (the positive set) and $f(x, y) < 0$ (the negative set). If the function divides space into the curve and two other connected area only (that is, any two points in a connected area may be joined by a curvilinear line which does not cross the curve), then these areas may be identified with the positive and negative sets defined by f. However, be wary, there are many elementary functions (for example, $g(x, y) \equiv \cos(y) - \sin(x)$) that define not one but a series of curves and hence divide space into possibly an infinite number of connected areas (note that $g(x, y) \equiv g(x + 2m\pi, y + 2n\pi$ for all integers m and n). So it is possible that two unconnected areas can both belong to the positive set.

Note that the functional representation need not be unique. We could have put the line into an equivalent form

$$f'(x, y) \equiv bx + c - ay$$

in which case the positive set of this function is the negative set of our original, and vice versa.

The case where the curve does divide space into two connected areas is very useful in computer graphics, as we shall see in the study of two-dimensional and (especially) three-dimensional graphics algorithms. Take for example the straight line

$$f(x, y) \equiv ay - bx - c$$

where a point (x_1, y_1) is on the same side of the line as (x_2, y_2) if and only if $f(x_1, y_1)$ has the same non-zero sign as $f(x_2, y_2)$. The functional representation tells us more about a point (x_1, y_1) than just on which side of a line it lies - it also enables us to calculate the distance of the point from the line.

Suppose we have the above line, then its direction vector is (a, b). A line perpendicular to this will have the direction vector $(-b, a)$. (Why? Because the product of the tangents of two mutually perpendicular lines is -1 ; see McCrae, 1953.) So the point $\boldsymbol{q}$ on the line closest to the point $\boldsymbol{p} \equiv (x_1, y_1)$ is of the form

$$q \equiv (x_1, y_1) + \mu(-b, a)$$

Therefore, a new line that joins p to q is perpendicular to the original line. Since $\boldsymbol{q}$ lies on this original line, then

$$f(q) = f(x_1, y_1) + \mu(-b, a)) = 0$$

that is

$$a \times (y_1 + \mu \times a) - b \times (x_1 - \mu \times b) - c = f(x_1, y_1) + \mu(a^2 + b^2) = 0$$

Hence

$$\mu = f(x_1, y_1)/(a^2 + b^2)$$

The point $\boldsymbol{q}$ is a distance $\mu \times |\ (-b, a)$ from (x_1, y_1) which naturally means that the distance of (x_l, y_1) from the line is $m \times \sqrt{}(a^2 + b^2) = -f(x_1, y_1)/\sqrt{}(a^2 + b^2)$: the sign denotes on which side of the line the point is lying. If $a^2 + b^2 = 1$ then $|\ f(x_1, y_1)\ |$ gives the distance of the point (x_1, y_1) from the line.

This idea leads us directly to a way of implementing *convex areas*; these areas are such that a straight line segment that joins any two points within the area lies totally inside the area. We shall limit our study to convex polygons, however, since it is obvious that any convex area may be approximated by a polygon, providing that it has enough sides.

Suppose we have a convex polygon with n vertices $\{\boldsymbol{p} \equiv (x_1, y_1) \mid i = 1, 2, \ldots, n\}$ taken in order around the polygon (either clockwise or anticlockwise) - we shall call such a description of a convex polygon an *oriented convex set* of vertices. The problem of finding whether such a set is clockwise or anticlockwise is considered in chapter 7. The n boundary edges of the polygon are segments of the lines

$$f_i(x, y) \equiv (x_{i+1} - x_1) - (y - y_1) - (y_{i+1} - y_\mathrm{i}) - (x - x_i)$$

where $i = 1, \ldots, n$, and the addition in the subscripts is modulo n (that is, $n + j \equiv j$ for $1 \leq \mathrm{j} \leq n$). Try to explain why these formulae do actually describe the line segments!

This systematic definition of the lines enables us to define the inside of the convex area. Any given line segment, say the one joining $\boldsymbol{p}i$ to $\boldsymbol{p}_{i+1}$ for some i, is such that the points inside the body must lie on the same side of this line as the remaining vertices of the polygon, in particular $\boldsymbol{p}_{i+2}$. So the inside is given by

$$\{(x, y) \mid \text{sign of } f_i(x, y) = \text{sign of } f_i(x_{i+2}, y_{i+2}) \neq 0 \text{: } i = 1, \ldots, n\}$$

A point on the boundary is given by

$$\begin{aligned}&\{(x, y) \mid \text{there exists one } j, \text{ or two if } (x, y) \text{ is a corner}\\ &\text{where } 1 \leq j \leq n \text{ such that } f_i(x, y) = 0 \text{ and}\\ &\text{sign of } f_i(x, y) = \text{sign of } f_i(x_{i+2}, y_{i+2}) \neq 0\text{: } i \neq j \text{ and } 1 \leq i \leq n\}\end{aligned}$$

A point outside the area is defined

$$\begin{aligned}&\{(x, y) \mid \text{there exists one } j,\ 1 \leq j \leq n \text{ such that}\\ &0 \neq \text{sign of } f(x, y) \neq \text{sign of } f_i(x_{j+2}, y_{\mathrm{i}+2}) \neq 0\}\end{aligned}$$

Naturally the additions of subscripts are all modulo n. This technique of ‘inside and outside’ is fundamental to the hidden surli' ce algorithm of chapter 12.

Example 3.3
Suppose we are given the convex polygon with vertices (1 , 0), (5, 2), (4, 4) and (–2, 1): see figure 3.4. 1n this order the wrtices obviously has an anticlockwise orientation. Are the points (3, 2), (1, 4), (3, 1) inside, outside or on the boundary of the polygon? What is the distance of (4, 4) from the first line?

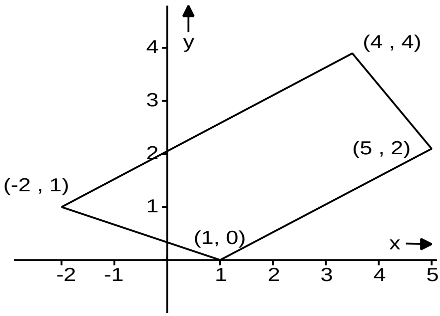

Figure 3.4

$$f_1(x, y) \equiv (5 - 1) \times (y - 0) - (2 - 0) \times (x - 1) \equiv 4y - 2x + 2$$

$$f_2(x, y) \equiv (4 - 5) \times (y - 2) - (4 - 2) \times (x - 5) \equiv -y - 2x + 12$$

$$f_3(x, y) \equiv (-2 - 4) \times (y - 4) - (1 - 4) \times (x - 4) \equiv -6y + 3x + 12$$

$$f_4(x, y) \equiv (1 + 2) \times (y - 1) - (0 - 1) \times (x + 2) \equiv 3y + x - 1$$

Hence point (3, 2) is inside the body because $f_1(3, 2) = 4$ and $f_1(4, 4) = 10$; $f_2(3, 2) = 4$ and $f_2(-2, 1) = 15$; $f_3(3, 2) = 9$ and $f_3(1, 0) = 15$; $f_4(3, 2) = 8$ and $f_4(5, 2) = 10$ - all with the saine positive signs.
Point (1, 4) is outside the body because $f_3(1, 4) = -9$ and $f_3(1 , 0) = 15$ – opposite signs.
Point (3, 1) is on the boundary because $f_1(3, 1) = 0$, $f_2(3, 1) = 5$, $f_2(3, 1) = 15$ and $f_4(3, 1) = 5$.

In fact there is no need to work out $f_i(x_{i+2}, y_{i+2})$ for every i - since they all have the same sign, once we have calculated $f_i(x3, y_3)$ then we can work with this value throughout.

(4, 4) is a distance $(f_1(4, 4)/\sqrt{(4^2 + 2^2)} = 10/\sqrt{20}) =,/5$ from line 1

Exercise 3.3

Imagine two convex polygons that intersect one another. The area of intersection is also a convex polygon. Use the methods that are mentioned in this chapter to calculate the wrtices of the new polygon.

Having dealt with the functional representation of a line, what about the parametric form? We noted that this form is one where the x-coordinate and y-coordinate of a general point on the curve are given in terms of parameter(s) (which could be the x or y values themselves), together with a range for the range for the parameter. So we have aheady seen a parametric form of a line, it is simply the base and directional representation

$$\begin{aligned} \boldsymbol{b} + \mu d &\equiv (x_1, y_1) + \mu(x_2, y_2) \\ &\equiv (x_1 + \mu \times x_2, y_1 + \mu \times y_2) \qquad \text{where } -\infty < \mu < \infty \end{aligned}$$

Here μ is the parameter, and $x_1 + \mu \times x_2$ and $y_1 + \mu \times y_2$ are the respective x and y values, which depend only on variable μ.

We can also produce functional representations and parametric forms for most well-behaved curves. For example a sine curve is given by $f(x, y) \equiv y - \sin(x)$ in functional representation, and by $(x, \sin(x))$ with $-\infty < x < \infty$ in its param' tric form. The general conic section (ellipse, parabola and hyperbola) is represented by the general function

$$f(x, y) \equiv a \times x_2 + b \times y_2 + h \times x \times y + f \times x + g \times y + c$$

where the coefficients a, b, c, f, g, h uniquely identify a curve. A circle centred at the, origin of radius has $a = b = 1$, $f = g = h = 0$ and $c = -r^2$, whence $f(x, y) \equiv x^2 + y^2 - r^2$. All the points (x, y) on the circle are such that $f(x, y) < 0$, and the outside of the circle $f(x, y) < 0$, and the outside of the circle $f(x, y) > 0$. The parametric form of this circle is $(r \times \cos \alpha, r \times \sin \alpha)$ where $0 \le \alpha \le 2\pi$. (We have already met the parametric form ofa circle, ellipse and spiral in chapter 2.)

It is very useful to experiment with these (and other) concepts in two dimensional geometry. There will be many occasions when it is necessary to include these ideas in programs, as well as the ever-present need when we are generating coordinate data for diagrams.

Example 3.4

Suppose we wish to draw a circular ball (radius r) that is disappearing down an elliptical hole (major axis a, minor axis b) - see figure 3.5. Parts of both the ellipse and circle are obscured.

Let the ellipse be centred on the origin with the major axis horizontal and the centre of the circle a distance d vertically above the origin. The ellipse has the functional representation

$$f_e(x, y) \equiv x^2/a^2 + y^2/b^2 - 1$$

and in parametric form

$$(a \times \cos \alpha, b \times \sin \alpha) \text{ with } 0 \le \alpha \le 2\pi$$

For the circle

$$f_c(x, y) \equiv x^2 + (y - d)^2 - r^2$$

and in parametric form

$$(r \times \cos \lambda, d + r \times \sin \lambda) \text{ where } 0 \le \lambda \le 2\pi$$

To generate the picture we must find the points (x, y) coniinon to the circle and ellipse (if any). As a useful demonstration we shall mix the representations in searching for a solution, by using the functional representation for the circle and the parametric form of the ellipse.

So we searching for the points $(x, y) \equiv (a \times \cos \alpha, b \times \sin \alpha)$ on the ellipse that also satisfy $f_c(x, y) = 0$. That is

$$a^2 \times \cos^2\alpha + (b \times \sin \alpha - d)^2 - r^2 = 0$$

or

$$a^2 \times \cos^2\alpha + b^2 \times \sin^2\alpha - 2 \times b \times d \times \sin \alpha + d^2 - r^2 = 0$$

And since cos2a =1 – sin2a, then

$$(b^2 - a^2) \times \sin^2\alpha - 2 \times b \times d \times \sin \alpha + a^2 + d^2 - r^2 = 0$$

This is a simple quadratic equation in the unknown sin a, which is easily solved (the quadratic equation $Ax^2 + Bx + C = 0$ has two roots given by $(-B \pm \sqrt{(B^2 - 4 \times A \times C)})/(2 \times A))$. For each value of sin a we can find values for a with $0 \le \alpha \le 2\pi$ (if they exist) and we can then cakulate the points of intersection $(a \times \cos \alpha, b \times \sin \alpha)$.

There is no hard and fast rule about which representittion to use in any given situation - a *feel* for the method is required and that only comes with experience.

Exercise 1.4
Write a program that will draw figure 3.5.

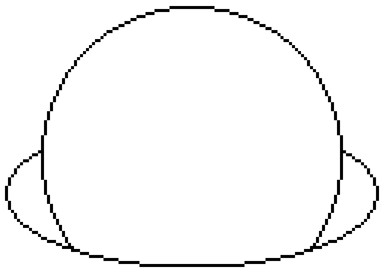

Figure 3.5

Complete Programs

I 'lib1' and listing 3.1 : no data required.

II Listing 3.2. Data required: four coordinate pairs (X1, Y1), (X2, Y2), (X3, Y3) and (X4, Y4).